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Some exact solutions for free convective flows over heated semi-infinite surfaces in porous media

D. A. S. Rees† and A. P. Bassom

Department of Mathematics, North Park Road, University of Exeter, Exeter, Devon EX4 4QE, U.K.

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1. INTRODUCTION

The study of convection generated by a heated semi-infinite surface embedded in a saturated porous medium has attracted extensive treatment in recent years. Of main concern has been the practical need to determine accurately the heat transferred into the porous medium from heated surfaces of various orientations. After the pioneering work of Cheng and Minkowycz [1] and Cheng and Chang [2], who considered flows generated by vertical and upward-facing horizontal surfaces, respectively, attention has been focused on higher order analyses (see refs. [3–6]). Detailed reviews of much of this work are given in Cheng [7] and Tien and Vafai [8]. However, the accuracy of high order analyses is limited due to the appearance of eigensolutions at some point in the expansion. This is due to the asymptotic nature of the analysis and a lack of precise knowledge of the effects of the leading edge. But we note in passing that a recent paper by Pop *et al.* [9] has sought to account for the 'leading-edge effect' by means of a deformed streamwise coordinate.

In this note we reconsider two of the more well-researched configurations. We consider a wedge-shaped region of saturated porous medium bounded by two semi-infinite surfaces, one heated isothermally, the other insulated. In particular, we study the two cases: (i) a vertical heated surface with a wedge angle of π, and (ii) a horizontal upward-facing surface with a wedge angle of $3\pi/2$. It is shown that, for these configurations, the full non-linear governing equations reduce to a set of ordinary differential equations upon introduction of appropriate coordinate transformations. These ODEs are, in fact, identical to those describing the classical leading order boundary layer profiles, and therefore detailed descriptions of the flow and temperature fields in the neighbourhood of the leading edge are determined, as are expressions for the heat transferred into the medium.

2. STATEMENT OF THE PROBLEM

The configuration we consider is as described above and shown in ref. [6]. The surface $y = 0$, $x > 0$ is held at a nondimensional temperature of unity whilst the ambient temperature of the saturated medium is zero (see ref. [6] for details of the nondimensionalization). Assuming that Darcy's law and the Boussinesq approximation are both valid, the two-dimensional equations become

$$\psi_{xx}+\psi_{yy} = (\cos\delta)\theta_y-(\sin\delta)\theta_x \tag{1a}$$

$$\theta_{xx}+\theta_{yy} = \psi_y\theta_x-\psi_x\theta_y. \tag{1b}$$

Since there is no natural length scale in the problem the Rayleigh number can be considered either to have been

† Permanent address: School of Mechanical Engineering, University of Bath, Claverton Down, Bath BA2 7AY, U.K.

NOMENCLATURE

f scaled non-dimensional streamfunction in Section 3
$\bar{f}$ scaled non-dimensional streamfunction in Section 4
g non-dimensional temperature in Section 3
$\bar{g}$ non-dimensional temperature in Section 4
q non-dimensional heat transfer coefficient
r non-dimensional radial coordinate
u non-dimensional free-stream velocity in Section 5
x non-dimensional coordinate along the heated surface
y non-dimensional coordinate perpendicular to the heated surface
X arbitrary value of x.

Greek symbols
α relative orientation of the two surfaces
δ inclination of the heated surface relative to the vertical
η modified similarity variable for the vertical configuration
$\bar{\eta}$ modified similarity variable for the horizontal configuration
θ non-dimensional temperature
ξ coordinate orthogonal to η
$\bar{\xi}$ coordinate orthogonal to $\bar{\eta}$
ϕ angular coordinate
ψ non-dimensional streamfunction.

scaled out of the equations or set to unity. In terms of polar coordinates $x = r\cos\phi$, $y = r\sin\phi$, the boundary conditions may be written conveniently as

$$\psi = 0, \quad \theta = 1 \quad \text{on} \quad \phi = 0$$

$$\psi = 0, \quad \frac{\partial\theta}{\partial\phi} = 0 \quad \text{on} \quad \phi = \alpha$$

$$\theta \to 0, \quad \psi = o(r) \quad \text{as} \quad r \to \infty, \quad 0 < \phi < \alpha. \tag{2}$$

3. ANALYSIS FOR THE VERTICAL SURFACE

A vertical heated surface corresponds to setting $\delta = 0$ in equation (1a). It is shown in ref. [6] that the similarity variable $\eta = y/x^{1/2}$ reduces the boundary layer approximation to equations (1) to a pair of ordinary differential equations the numerical solution of which is easily effected. The solution to the full problem is determined by transforming to parabolic coordinates (ξ,η) given by

$$x = \tfrac{1}{4}(\xi^2 - \eta^2), \quad y = \frac{\xi\eta}{2}. \tag{3}$$

In view of the analysis in Section 4 it is worth noting that, in terms of polar coordinates, we have the alternative representation

$$\xi = 2r^{1/2}\cos\frac{\phi}{2}, \quad \eta = 2r^{1/2}\sin\frac{\phi}{2}. \tag{4}$$

Then, setting $\delta = 0$, equations (1) transform to

$$\psi_{\xi\xi} + \psi_{\eta\eta} = \tfrac{1}{2}(\eta\theta_\xi + \xi\theta_\eta) \tag{5a}$$

$$\theta_{\xi\xi} + \theta_{\eta\eta} = \psi_\eta\theta_\xi - \psi_\xi\theta_\eta. \tag{5b}$$

Although the governing equations are not simplified by means of this transformation, they may be seen to admit the exact solution

$$\psi = \tfrac{1}{2}\xi f(\eta), \quad \theta = g(\eta) \tag{6}$$

if $f(\eta)$ and $g(\eta)$ satisfy the equations

$$f'' = g', \quad g'' + \tfrac{1}{2}fg' = 0 \tag{7a,b}$$

subject to

$$f(0) = 0, \quad g(0) = 1, \quad f' \to 0, \quad g \to 0 \quad \text{as} \quad \eta \to \infty. \tag{7c}$$

These equations and boundary conditions are identical to those defining the leading-order boundary layer profile in ref. [1]. However, (6) satisfies all the boundary conditions given in (2) only if α, the wedge angle, is equal to π. We note that (6) also satisfies the appropriate equations and boundary conditions for an isolated semi-infinite heated surface in an unbounded medium.

Plots of the streamlines and isotherms are displayed in Figs. 1 and 2, respectively, where the formation of the momentum and thermal boundary layers may be seen as x increases. The depth of penetration of the temperature field into the region below the leading edge is also clearly evident.

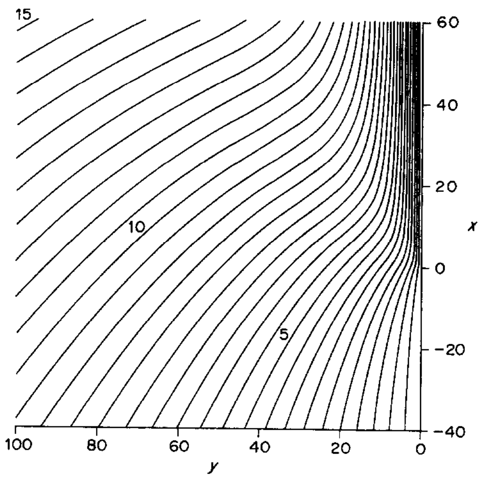

FIG. 1. Streamlines for the vertical configuration plotted with an interval of 0.5 between adjacent contours.

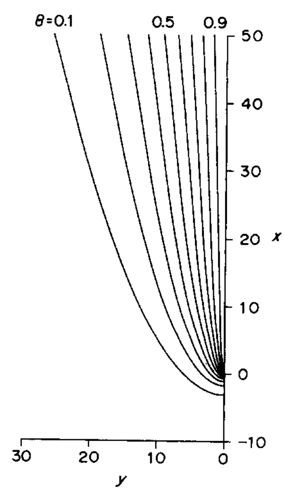

FIG. 2. Isotherms for the vertical configuration plotted with an interval of 0.1 between adjacent contours.

The heat transferred from the plate between $x = 0$ and X is given precisely by

$$q = -\int_0^X \frac{\partial \theta}{\partial y}\bigg|_{y=0} \mathrm{d}x = -2X^{1/2} f''(0) \tag{8}$$

where $f(0) = -0.44374832$. For $x \gg y$ the usual boundary layer profile $\psi \sim x^{1/2} f(\eta)$, $\theta \sim g(\eta)$ is recovered, where $\eta \sim y x^{-1/2}$, whilst the flow well away from the boundary layer and the leading edge is $\psi \sim r^{1/2} \cos(\phi/2) f(\infty)$, where $f(\infty) = 1.61612544$ and θ is exponentially small.

4. ANALYSIS FOR THE HORIZONTAL SURFACE

For the horizontal upward-facing surface the correct coordinate transformation is obtained by first considering the equations in polar coordinates. It may be shown that equations (1) with $\delta = \pi/2$ are satisfied by

$$\psi = \tfrac{1}{3}\bar{\xi} \bar{f}(\bar{\eta}), \quad \theta = \bar{g}(\bar{\eta}) \tag{9a,b}$$

where

$$\bar{\xi} = 3r^{1/3} \cos\frac{\phi}{3}, \quad \bar{\eta} = 3r^{1/3} \sin\frac{\phi}{3} \tag{10a}$$

(cf. the form of (4) for the vertical configuration) or, alternatively

$$x = \frac{\bar{\xi}}{27}(\bar{\xi}^2 - 3\bar{\eta}^2), \quad y = \frac{\bar{\eta}}{27}(3\bar{\xi}^2 - \bar{\eta}^2) \tag{10b}$$

provided that $\alpha = 3\pi/2$. Here $\bar{f}$ and $\bar{g}$ satisfy

$$\bar{f}'' - \tfrac{2}{3}\bar{\eta}\bar{g}' = 0, \quad \bar{g}'' + \tfrac{1}{3}\bar{f}\bar{g}' = 0 \tag{11a,b}$$

subject to

$$\bar{f}(0) = 0, \quad \bar{g}(0) = 1, \quad \bar{f}' \to 0, \quad \bar{g} \to 0 \quad \text{as} \quad \bar{\eta} \to \infty. \tag{11c}$$

This system was first given in ref. [2] to describe the leading-order boundary layer profile over a horizontal surface. Thus (9) satisfies the full equations in the whole flow region.

Plots of the streamlines and isotherms are displayed in Figs. 3 and 4, respectively. Once more the formation of the boundary layer and the effect of the leading edge can be seen. The heat transferred from the surface between $x = 0$ and X is given by

$$q = -\int_0^X \frac{\partial \theta}{\partial y}\bigg|_{y=0} \mathrm{d}x = -3X^{1/3}\bar{g}'(0) \tag{12}$$

where $\bar{g}'(0) = -0.43021314$. For large values of x the boundary layer profile is recovered: $\psi \sim r^{1/3} \bar{f}(\bar{\eta})$, $\theta \sim \bar{g}(\bar{\eta})$ where $\bar{\eta} \sim y/x^{2/3}$. Far from the boundary layer and the leading edge θ is exponentially small and $\psi \sim r^{1/3} \cos(\theta/3)\bar{f}(\infty)$ where $\bar{f}(\infty) = 2.81581039$.

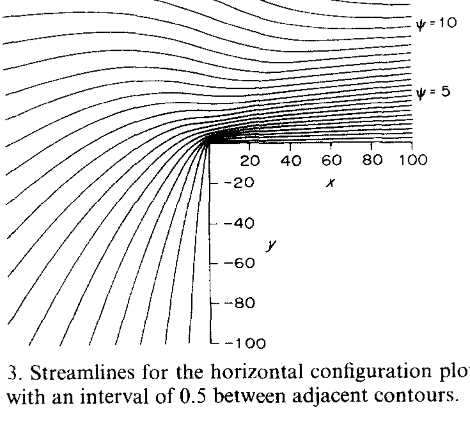

FIG. 3. Streamlines for the horizontal configuration plotted with an interval of 0.5 between adjacent contours.

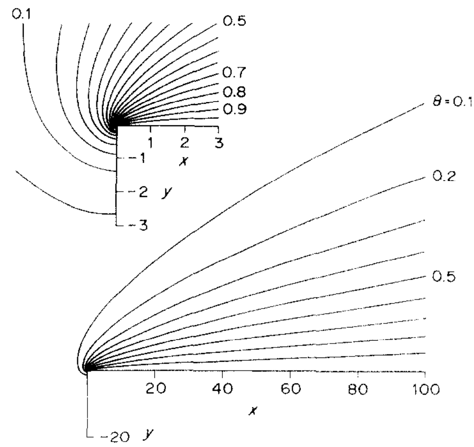

FIG. 4. Isotherms for the horizontal configuration plotted with an interval of 0.1 between adjacent contours. Inset: a close-up diagram of the isotherms in the corner region with an interval of 0.05 between adjacent contours.

5. DISCUSSION

Two specific cases of free convection from isothermal surfaces in porous media have been considered and shown to admit solutions expressible in terms of the solutions to ordinary differential equations. Precise expressions for the heat transferred into the medium have been presented.

As these solutions are valid in the whole of the flow region rather than asymptotically for large distances from the leading edge, we can investigate the detailed structure of the flow near the leading edge. For the vertical surface

$$\psi = r\sin\phi + O(r^{3/2}),$$

$$\theta = 1 + 2r^{1/2}\sin\left(\frac{\phi}{2}\right)g'(0) + O(r) \tag{13a,b}$$

for small r, whilst for the horizontal surface

$$\psi = \frac{3}{2}r^{2/3}\sin\left(\frac{2\phi}{3}\right)\bar{f}'(0) + O(r),$$

$$\theta = 1 + 3r^{1/3}\sin\left(\frac{\phi}{3}\right)\bar{g}'(0) + O(r^{2/3}) \tag{14}$$

where $\bar{f}'(0) = 1.05574767$. The flow near the leading edge of the vertical surface (see equation (13a)) is uniformly upwards, to leading order, as $\psi \sim y$. As regards the other expressions, equations (13b) and (14), although they remain finite as $r \to 0$, their derivatives with respect to r become unbounded. Doubt must therefore be cast on the degree of idealization inherent in the mathematical modelling of the physical problem, namely the assumption of a perfectly conducting heated surface. By continuity we expect that a similar doubt should be expressed for other inclinations δ, and wedge angles α. However, we stress that, given the idealization, the presented solutions solve the full non-linear equations.

There exists at least one example of a mixed convection problem amenable to the approach presented here. Consider the effect of uniform vertical pressure gradient on the free convection problem described in Section 2. Thus we solve equations (1) subject to the boundary conditions

$$\psi = 0, \quad \theta = 1 \quad \text{on} \quad \phi = 0 \tag{15a}$$

$$\psi = 0, \quad \frac{\partial \theta}{\partial \phi} = 0 \quad \text{on} \quad \phi = \pi \tag{15b}$$

$$\theta \to 0, \quad \psi \to ur\sin\phi \quad \text{as} \quad r \to \infty. \tag{15c}$$

Here u is a non-dimensional measure of the pressure-induced vertical velocity. It may be shown that ψ and θ are given by

$$\psi = \tfrac{1}{2}\xi f(\eta) + \tfrac{1}{2}\xi\eta u, \quad \theta = g(\eta) \tag{16}$$

where

$$f'' = g', \quad g'' + \tfrac{1}{2}(f + u\eta)g' = 0 \tag{17a,b}$$

subject to

$$f(0) = 0, \quad g(0) = 1, \quad f', g \to 0 \quad \text{as} \quad \eta \to \infty \tag{17c}$$

and where (ξ, η) is as defined in Section 3. When $u = 0$ we recover the analysis of Section 3 and f' and g decay exponentially as η increases. For positive values of u the decay is superexponential, both functions being proportional to $\exp(-u\eta^2)$ to leading order. For negative values of u the only steady-state solution is, of course, $\psi = uy$ with $\theta = 1$.

There is another configuration cited in the literature dealing with flow in a saturated porous wedge where the governing partial differential equations can be reduced to a pair of ordinary differential equations. In ref. [11] an analysis of flow in a wedge is considered where the bounding surfaces are held at temperatures which are inversely proportional to r, the distance from the apex. Unfortunately we have not found any other 'boundary layer' problems which may be reduced to a set of ordinary differential equations. In this regard we have considered the following problems: arbitrarily inclined surfaces in a porous medium, similar configurations for the analogous problem of a semi-infinite surface immersed in a Newtonian fluid and semi-infinite surfaces with a power-law temperature distribution. Nevertheless, it should be possible to obtain numerical solutions of some of these problems using coordinate transformations similar to those used here since the shape of the boundary layer is incorporated into the transformation. We hope to report on this in the future.

Recent work on the instability of thermal boundary layers in porous media (see, e.g. refs. [12, 13]) uses boundary layer theory to approximate the basic flow. Since it is shown in ref. [12] that the boundary layer flow from a vertical surface is stable we would expect the same conclusion to apply if the exact solution presented here were to be used as the base flow. However, flow from a horizontal surface is inherently unstable (see ref. [13]). Therefore, it should be possible to determine more accurately where the flow becomes unstable, at least for the case of a wedge angle of $3\pi/2$, and to calculate the wavelength of the vortices. Again, it is hoped to report on this at some point.

It is worthwhile, in conclusion, to consider this work in the wider context of boundary layer theory. Kaplun, in his seminal paper [14] on the use of optimal coordinates for boundary layer flows gave a method for determining such coordinate systems. The transformations used here may therefore be regarded, albeit fortuitously, as the ultimate in this respect as they yield complete information about the flow field. In general, in high order boundary layer theory, one encounters eigensolutions and logarithmic terms. For the present configurations it is shown in ref. [6] that there are no logarithmic terms corresponding to the first eigensolution. The existence of the exact solutions presented here implies that the coefficients of all eigensolutions can be determined.

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